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A lower bound for the canonical height on elliptic curves over abelian extensions[☆]

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Abstract

Let E/K be an elliptic curve defined over a number field, let \hat{h} be the canonical height on E , and let K^{ab}/K be the maximal abelian extension of K . Extending work of M. Baker (IMRN 29 (2003) 1571–1582), we prove that there is a constant $C(E/K) > 0$ so that every nontorsion point $P \in E(K^{\text{ab}})$ satisfies $\hat{h}(P) > C(E/K)$.

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1. Introduction

The classical Lehmer conjecture [14] asserts that there is an absolute constant $C > 0$ so that any algebraic number α that is not a root of unity satisfies $h(\alpha) > C/[\mathbb{Q}(\alpha) : \mathbb{Q}]$. Recently Amoroso and Dvornicich [1] showed that if α is restricted to lie in \mathbb{Q}^{ab} , then the stronger inequality $h(\alpha) > C$ is true. The analog of Lehmer's conjecture for elliptic curves and abelian varieties has also been much studied [3, 7, 11, 13, 16, 17, 26]. Baker [4] has proven the elliptic analog of the Amoroso–Dvornicich result if the elliptic curve either has complex multiplication or has non-integral j -invariant, but he was unable to handle the general case of integral j -invariant. In this note we extend Baker's result to include all elliptic curves.

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Theorem 1. *Let K/\mathbb{Q} be a number field, let E/K be an elliptic curve, and let $\hat{h}: E(\bar{K}) \rightarrow \mathbb{R}$ be the canonical height on E . There is a constant $C(E/K) > 0$ such that every nontorsion point $P \in E(K^{\text{ab}})$ satisfies*

$$\hat{h}(P) > C.$$

Remark 1. This theorem gives a proof of Baker’s Conjecture 1.10 [4] for elliptic curves. In Baker’s terminology, we will prove that if E does not have complex multiplication, then $E(K^{\text{ab}})$ has the “strong discreteness property.”

We begin in Section 2 with a brief history of known results on Lehmer’s conjecture. In Section 3 we set notation and recall basic facts about local heights. Section 4 discusses torsion points and quotes a result of Serre that will be needed in the proof of Theorem 1. In Section 5 we take up the unramified case. Here we use the characteristic polynomial of Frobenius at a prime ideal \mathfrak{P} , evaluated at Frobenius, to annihilate points modulo \mathfrak{P} . This device replaces the use of a complex multiplication map in [4], allowing us to deal with general elliptic curves. Next, in Section 6, we deal with the ramified case. We simplify the argument in [4] by applying $\tau - 1$ twice to our point, the first time to move it into the formal group, and the second time to make it a difference of conjugate points in the formal group. Finally, in Section 7, we complete the proof of our main theorem. In Section 8, we indicate how many of the arguments can be generalized to the case of abelian varieties. In particular, we sketch a proof that $\hat{h}(P) > C(A/K)$ for all nontorsion points $P \in A(K^{\text{ab}})$ if the abelian variety A/K has no complex multiplication abelian subvarieties.

2. Earlier results on Lehmer’s Conjecture

For the convenience of the reader, in this section we summarize some of the known results regarding Lehmer’s Conjecture for number fields and for elliptic curves. Detailed information and an extensive bibliography for the former is given on the Lehmer Conjecture Website [15].

2.1. The classical Lehmer Conjecture

Let $\alpha \in \bar{\mathbb{Q}}^*$, assume that α is not a root of unity, and let $D = [\mathbb{Q}(\alpha) : \mathbb{Q}]$. The *Lehmer Conjecture* says that there is a constant $c > 0$ so that $h(\alpha) \geq cD^{-1}$ for all such α .

Blanksby and Montgomery [5] and Stewart [31] independently, and by rather different methods, proved that

$$h(\alpha) \geq cD^{-1}(\log D)^{-1}.$$

This was superceded by Dobrowolski's estimate [7]

$$h(\alpha) \geq cD^{-1}(\log \log(D)/\log(D))^3,$$

which is currently the best known general result. In the special case that α^{-1} is not a Galois conjugate of α (one says that α is nonreciprocal), Smyth [30] proved the full Lehmer Conjecture $h(\alpha) \geq cD^{-1}$. See also [1] for an alternative proof of Smyth's result.

2.2. Lehmer's Conjecture for elliptic curves

Let E/K be an elliptic curve, let \hat{h} be the canonical height on E , let $P \in E(\bar{K})$ be a nontorsion point, and let $D = [K(P) : K]$ be the degree of the field of definition of P . The elliptic analog of Lehmer's Conjecture says that $\hat{h}(P) \geq cD^{-1}$. Table 1 summarizes the history of lower bounds for $\hat{h}(P)$, where c denotes a positive constant that depends on E/K , but not on P . We also note that Pacheco [20, Theorem 2.3] has proven the elliptic Lehmer Conjecture for function fields over finite fields.

2.3. Lehmer's Conjecture in abelian extensions

In the classical setting, Amoroso and Dvornicich [1] recently gave a “super-Lehmer” lower bound for every $\alpha \in \mathbb{Q}^{\text{ab}}$ that is not a root of unity:

$$h(\alpha) \geq c.$$

Amoroso and Zannier [2] generalize this to relative abelian extensions K^{ab}/K . More generally, they give a Dobrowolski-type bound for $\alpha \in \bar{K}$ with the degree $D = [K(\alpha) : K]$ replaced by the “nonabelian” part of the degree $D' = [K(\alpha) : K(\alpha) \cap K^{\text{ab}}]$.

Table 2 gives the history of lower bounds for the canonical height in $E(K^{\text{ab}})$ for elliptic curves E/K . In particular, the results in the present paper complete the one case left undone in [4], namely non-CM elliptic curves with integral j -invariant. We observe that the non-CM integral j -invariant case often presents the greatest difficulties when studying algebraic points on elliptic curves, see for example Serre's

Table 1

History of lower bounds for \hat{h} in $E(\bar{K})$

$\hat{h}(P) \geq$	Restriction on E	Reference
$cD^{-10}(\log D)^{-6}$	none	Anderson–Masser (1980) [3]
$cD^{-1}(\frac{\log \log(D)}{\log(D)})^3$	CM	Laurent (1983) [13]
$cD^{-3}(\log D)^{-2}$	none	Masser (1989) [17]
$cD^{-2}(\log D)^{-2}$	j nonintegral	Hindry–Silverman (1990) [11]

Table 2

History of lower bounds for \hat{h} in $E(K^{\text{ab}})$

$\hat{h}(P) \geq$	Restriction on E	Reference
cD^{-2}	none	Silverman (1981) [26]
$cD^{-1}(\log D)^{-2}$	none	Masser (1989) [17]
$cD^{-2/3}$	j nonintegral	Hindry–Silverman (1990) [11]
c	j nonintegral or CM	Baker (2002) [4]
c	none	Silverman (2003)

theorem [22] on the image of Galois in $\text{End}(E_{\text{tors}})$. The reason is that if E has CM or if E has nonintegral j -invariant, then there is some control over the action of Galois on torsion. In the former case, there is almost complete control via class field theory, and in the latter case, the Tate curve over $K_{\mathfrak{p}}$ gives control for the decomposition group at \mathfrak{p} .

2.4. Lehmer's Conjecture for abelian varieties

A number of authors have considered the analog of Lehmer's Conjecture for abelian varieties A/K of dimension $g \geq 2$. Let \mathcal{L} be an ample symmetric line bundle on A , let $P \in A(\bar{K})$ be a nontorsion point, and let $D = [K(P) : K]$. Masser [16] proves that there is a constant $\kappa = \kappa(\dim A)$ so that

$$\hat{h}_{\mathcal{L}}(P) \geq cD^{-\kappa}.$$

If A has complex multiplication, then David and Hindry [6] generalize Laurent's result to give a Dobrowski-type estimate

$$\hat{h}(P) \geq cD^{-1}(\log \log(D)/\log(D))^{\kappa}.$$

Restricting to special types of fields, S. Zhang notes that the equidistribution theorems proven in [32,34] imply that if L is a finite extension of a totally real field, then $A(L)_{\text{tors}}$ is finite and $\hat{h}_{\mathcal{L}}(P) \geq c$ for all nontorsion $P \in A(L)$. (See [4, Theorem 1.8].) In particular, this is true for $A(K^{\text{ab}})$ if K itself is totally real. [The author would like to thank Michael Rosen for pointing out this last fact.]

3. Preliminaries

In this section we set notation and recall some basic facts about the decomposition of the canonical height into a sum of local heights.

3.1. Notation

We set the following notation.

K/\mathbb{Q} a number field.

M_K the set of absolute values on a number field K extending the usual absolute values on \mathbb{Q} . We write $v(x) = -\log|x|_v$, and for nonarchimedean v , we denote the associated prime ideal by \mathfrak{p}_v .

E/K an elliptic curve defined over K .

\hat{h} the canonical height $\hat{h} : E(\bar{K}) \rightarrow \mathbb{R}$ on E , see [27, VIII, Section 9].

$\hat{\lambda}$ the local canonical height (Néron function)

$\hat{\lambda} : M_{\bar{K}} \times E(\bar{K}) \rightarrow \mathbb{R} \cup \infty$,
normalized as described in [29, VI.1.1].

Remark 2. With absolute values normalized as above, the product formula reads

$$\sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} v(\alpha) = 0 \quad \text{for all } \alpha \in K^*$$

We also have the usual formula for finite extensions L/K ,

$$\sum_{w \in M_L, w|v} [L_w : K_v] = [L : K].$$

Remark 3. If v is nonarchimedean, say corresponding to a prime ideal \mathfrak{p} of residue characteristic p and ramification index $e = e(\mathfrak{p}/p)$, then v is discrete and its smallest positive value is

$$\inf\{v(\alpha) : \alpha \in K^* \text{ and } v(\alpha) > 0\} = \frac{\log p}{e}.$$

This is clear, since if π is a uniformizer in K for \mathfrak{p} , then locally $(\pi)^e = (p)$, so $v(\pi) = v(p)/e = (\log p)/e$.

3.2. Local height functions

We recall some well-known properties of the local canonical height on an elliptic curve.

Theorem 2. *Let*

$$\hat{\lambda} : M_{\bar{K}} \times E(\bar{K}) \rightarrow \mathbb{R} \cup \{\infty\},$$

be the local canonical height (also called a Néron function) for the divisor (O) and normalized as described in [29, VI.1.1].

(a) For any finite extension L/K and any point $P \in A(L) \setminus \{O\}$,

$$\hat{h}(P) = \sum_{w \in M_L} \frac{[L_w : \mathbb{Q}_w]}{[L : \mathbb{Q}]} \hat{\lambda}_w(P).$$

(b) There is a constant $c_1(E/K) \geq 0$ so that for any finite extension L/K and any point $P \in A(L) \setminus \{O\}$,

$$\sum_{w \in M_L} \frac{[L_w : \mathbb{Q}_w]}{[L : \mathbb{Q}]} \min\{\hat{\lambda}_w(P), 0\} \geq -c_1(E/K).$$

(c) Let $v \in M_K$ be a finite place of good reduction for E , and choose a minimal Weierstrass equation for E at v ,

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Let L/K be a finite extension and let $w \in M_L$ lie over v . Then

$$\hat{\lambda}_w(P) = \max\{w(x/y), 0\} \quad \text{for all } P = (x, y) \in E(L) \setminus \{O\}.$$

Proof. (a) This is [29, Theorem VI.2.1].

(c) Since we are assuming that E has good reduction and that we have chosen a minimal Weierstrass equation, this is [29, Theorem VI.4.1], except that we have written $w(x/y)$ instead of $\frac{1}{2}w(x^{-1})$. (Note that since E has good reduction and the equation is minimal at v , the same is true for any extension w of v , and hence $w(\Delta) = 0$ and $E_0(L_w) = E(L_w)$.) The integrality of the Weierstrass equation implies easily that

$$w(x^{-1}) < 0 \Leftrightarrow w(x/y) < 0 \quad \text{and} \quad \min\{0, 3w(x)\} = \min\{0, 2w(y)\},$$

so $\frac{1}{2}w(x^{-1}) = w(x/y)$ if either is negative.

(b) Let $S_K \subset M_K$ be the union of the archimedean places of K and the set of places of K at which E has bad reduction, and let $S_L \subset M_L$ be the set of places of L lying above places of K . From (c), we know that $\hat{\lambda}_w(P) \geq 0$ for all $w \notin S_L$.

Suppose now that $v \in S_K$ is archimedean. We fix an isomorphism $E(\bar{K}_v) \cong E(\mathbb{C}) \cong \mathbb{C}/q_v^{\mathbb{Z}}$ with $0 < |q_v|_v < e^{-\pi}$. Then the local height is given by the explicit formula [29, VI.3.4]

$$\begin{aligned} \hat{\lambda}_v(u) = & \frac{1}{2} \mathbb{B}_2 \left(\frac{\log |u|_v}{\log |q_v|_v} \right) \log |q_v^{-1}|_v \\ & - \log |1 - u|_v - \sum_{n \geq 1} \log |(1 - q_v^n u)(1 - q_v^n / u)|_v, \end{aligned} \quad (1)$$

where $\mathbb{B}_2(t)$ is the Bernoulli polynomial $t^2 - t + \frac{1}{6}$ for $0 \leq t \leq 1$, extended periodically to \mathbb{R}/\mathbb{Z} . It is easy to see from (1) that

$$\hat{\lambda}_v(u) \geq \frac{1}{2} \mathbb{B}_2\left(\frac{\log|u|_v}{\log|q_v|_v}\right) \log|q_v^{-1}|_v - c_2$$

for an absolute constant c_2 . The polynomial $\mathbb{B}_2(t)$ has a minimum at $t = \frac{1}{2}$, so we find that

$$\hat{\lambda}_v(P) \geq -\frac{1}{24} \log|q_v^{-1}|_v - c_2 \quad \text{for all } P \in E(\bar{K}_v) \setminus \{O\}.$$

Since K and E are fixed, this says that there is a constant $c_3 = c_3(E/K)$ so that $\hat{\lambda}_v(P) \geq -c_3$ for all archimedean places $v \in M_K$ and all points $P \in E(\bar{K}_v) \setminus \{O\}$. It follows that for any finite extension L/K and any point $P \in A(L) \setminus \{O\}$,

$$\sum_{w \in M_L^\infty} \frac{[L_w : \mathbb{Q}_w]}{[L : \mathbb{Q}]} \min\{\hat{\lambda}_w(P), 0\} \geq \sum_{w \in M_L^\infty} \frac{[L_w : \mathbb{Q}_w]}{[L : \mathbb{Q}]} (-c_3) = -c_3.$$

Note that if w lies over v , then $\bar{L}_w \cong \bar{K}_v$, so we can compute the local height $\hat{\lambda}_w(P)$ using the inclusions $E(L) \subset E(L_w) \subset E(\bar{K}_v)$.

Next suppose that $v \in M_K$ is a nonarchimedean place at which E/K has split multiplicative reduction. Then the exact same argument gives an analogous lower bound for $\hat{\lambda}_v$, because the local height is given by the same formula (1), see [29, VI.4.2].

Finally, if E/K has additive or nonsplit multiplicative reduction at v , then we may either make a finite extension of K to reduce to one of the previous cases or use explicit formulas in these cases (see, e.g., [25, Table 1]) to obtain the desired lower bound. \square

Remark 4. Using explicit formulas for the local canonical height, it is not difficult to give an explicit estimate for the constant $c_1(E/K)$ in terms of the j -invariant and minimal discriminant $\mathcal{D}_{E/K}$ of E/K . Roughly, one can take c_1 to have the form

$$c_1 = c'_1 \max\{1, h(j), \log N_{K/\mathbb{Q}} \mathcal{D}_{E/K}\},$$

where c'_1 is an absolute constant. See [28] for a similar computation of explicit constants associated to local height functions.

Remark 5. We have chosen to prove Theorem 2(b) by appealing to explicit formulas for the local height on elliptic curves. It is also possible to prove this result by appealing to the M_K -continuity of the local height and using an M_K -compactness type of argument. Of course, $E(\bar{K}_v)$ is not even locally compact for nonarchimedean places v , so one must be careful. A detailed development of the correct concept, which is called M_K -boundedness, is given in [12].

4. Torsion points in abelian extensions

The following result is a simple consequence of a deep theorem of Serre.

Theorem 3 (Serre [22]). *Let K/\mathbb{Q} be a number field and let E/K be an elliptic curve without CM, that is, $\text{End}(E/\bar{K}) = \mathbb{Z}$. Then*

$$E(K^{\text{ab}})_{\text{tors}} \text{ is finite.}$$

In particular, $E(K^{\text{ab}})[\ell] = 0$ for all but finitely many primes ℓ .

Proof. Serre [22] proves that there is a finite set of primes S so that for all $\ell \notin S$, the Galois representation

$$\rho_\ell : G_{\bar{K}/K} \rightarrow \text{Aut}(E[\ell]) \cong \text{GL}_2(\mathbb{F}_\ell)$$

is surjective. Suppose that $T \in E(K^{\text{ab}})[\ell]$ with $T \neq O$ for some $\ell \notin S$. The group $\text{GL}_2(\mathbb{F}_\ell)$ acts transitively on the nonzero vectors in \mathbb{F}_ℓ^2 , so Serre's theorem tells us that the Galois orbit of T consists of all nonzero elements of $E[\ell]$. The conjugates of T are all defined over K^{ab} , so we conclude that $E[\ell] \subset E(K^{\text{ab}})$. Hence the representation ρ_ℓ factors through the maximal abelian quotient $G_{K^{\text{ab}}/K}$ of $G_{\bar{K}/K}$. This contradicts the fact that the image of ρ_ℓ is a nonabelian group, which completes the proof that $E(K^{\text{ab}})[\ell] = 0$ for all $\ell \notin S$. This statement suffices for our later applications. In order to prove the stronger statement that $E(K^{\text{ab}})_{\text{tors}}$ is finite, it remains to show that for any particular ℓ , the ℓ -power torsion in $E(K^{\text{ab}})$ is finite. This follows by a similar argument using the fact, also due to Serre [23], that the image of $G_{\bar{K}/K}$ in $\text{Aut}(T_\ell(E))$ is open, and thus has finite index. \square

5. The unramified case

In this section we prove the basic estimate required for the proof of our main result in the case that the extension L/K is unramified at a (small) prime \mathfrak{p} of K . We begin with the observation that there is an element of the group ring $\mathbb{Z}[G_{\bar{K}/K}]$ that simultaneously annihilates $E(K^{\text{ab}})$ modulo every prime of K^{ab} lying above \mathfrak{p} .

Theorem 4. *Let K/\mathbb{Q} be a number field, let \mathfrak{p} be a prime of K , and let E/K be an elliptic curve with good reduction at \mathfrak{p} . Let*

$$\Phi_{\mathfrak{p}}(X) = \det(X - \text{Frob}_{\mathfrak{p}}|T_\ell(E)) = X^2 - aX + q$$

be the characteristic polynomial of Frobenius at \mathfrak{p} acting on the ℓ -adic Tate module for some prime ℓ prime to \mathfrak{p} . Thus $\Phi_{\mathfrak{p}}(X) \in \mathbb{Z}[X]$ with $q = N_{K/\mathbb{Q}}\mathfrak{p}$ and $|a| \leq 2\sqrt{q}$ (see [27, Chapter V]).

Let $\bar{\mathfrak{p}}$ be a prime of \bar{K} lying over \mathfrak{p} and let $\sigma \in (\bar{\mathfrak{p}}, \bar{K}/K) \subset G_{\bar{K}/K}$ be in the associated Frobenius conjugacy class.

(a) For all $P \in E(\bar{K})$,

$$\Phi_{\mathfrak{p}}(\sigma)P \equiv O \pmod{\bar{\mathfrak{p}}}.$$

(This congruence is taking place on the Néron model of E , or more prosaically, on a Weierstrass equation for E that is minimal at \mathfrak{p} .)

(b) If $P \in E(\bar{K})$ satisfies $\Phi_{\mathfrak{p}}(\sigma)P = O$, then P is a torsion point.

Proof. (a) When reduced modulo $\bar{\mathfrak{p}}$, the element $\sigma \in G_{\bar{K}/K}$ acts as the q -power Frobenius map $f_q \in \text{End}(\tilde{E}/\mathbb{F}_{\bar{\mathfrak{p}}})$. Further, the map $\Phi_{\mathfrak{p}}(f_q)$ annihilates $T_{\ell}(\tilde{E}/\mathbb{F}_{\bar{\mathfrak{p}}})$, since $\Phi_{\mathfrak{p}}$ is the characteristic polynomial of f_q acting on $T_{\ell}(\tilde{E}/\mathbb{F}_{\bar{\mathfrak{p}}})$ and the Cayley–Hamilton theorem tells us that a linear transformation satisfies its own characteristic equation. However, we have the general fact that the map

$$\text{End}(E) \rightarrow \text{End}(T_{\ell}(E))$$

is injective (cf. [27, V.7.3]), so we conclude that $\Phi_{\mathfrak{p}}(f_q) = 0$ as an element of $\text{End}(\tilde{E}/\mathbb{F}_{\bar{\mathfrak{p}}})$. In other words,

$$\Phi_{\mathfrak{p}}(f_q)Q = O \quad \text{for all } Q \in \tilde{E}(\bar{\mathbb{F}}_{\bar{\mathfrak{p}}}).$$

Finally, using the fact that the reduction map commutes with the action of Galois, we see that for any $P \in E(\bar{K})$, the point $\Phi_{\mathfrak{p}}(\sigma)P$ is in the kernel of reduction. In other words, $\Phi_{\mathfrak{p}}(\sigma)P \equiv O \pmod{\bar{\mathfrak{p}}}$, which completes the proof of (a).

(b) Let $P \in E(\bar{K})$ satisfy $\Phi_{\mathfrak{p}}(\sigma)P = O$. Fix a finite Galois extension L/K with $P \in E(L)$, say of degree $m = [L : K]$. Then $\sigma^m = 1$ in $G_{L/K}$, so in particular, $\sigma^m P = P$. Let

$$r = \text{Resultant}(\Phi_{\mathfrak{p}}(X), X^m - 1) \in \mathbb{Z}.$$

The complex roots of $X^m - 1$ have absolute value 1 and the complex roots of $\Phi_{\mathfrak{p}}(X)$ have absolute value \sqrt{q} , so they have no complex roots in common. It follows that $r \neq 0$.

The resultant of two polynomials in $\mathbb{Z}[X]$ is a generator for the ideal that they generate, so we can find $a(X), b(X) \in \mathbb{Z}[X]$ satisfying

$$a(X)\Phi_{\mathfrak{p}}(X) + b(X)(X^m - 1) = r.$$

Substituting $X = \sigma$ gives the identity

$$a(\sigma)\Phi_{\mathfrak{p}}(\sigma) + b(\sigma)(\sigma^m - 1) = r$$

in the group ring $\mathbb{Z}[G_{\bar{K}/K}]$. Hence

$$rP = a(\sigma)(\Phi_{\mathfrak{p}}(\sigma)P) + b(\sigma)((\sigma^m - 1)P) = O,$$

so P is a point of finite order. \square

Corollary 5. *Continuing with the notation and assumptions from Theorem 4, let $P \in E(\bar{K})$ be a nontorsion point, and let e be the absolute ramification index of $\bar{\mathfrak{p}}$ in the field of definition $K(P)$ of P . (That is, let e be the ramification index of the prime ideal $\bar{\mathfrak{p}} \cap K(P)$ over the prime p .) Then*

$$\hat{\lambda}_{\bar{\mathfrak{p}}}(\Phi_{\mathfrak{p}}(\sigma)P) \geq \frac{\log p}{e}.$$

Proof. To ease notation, let $Q = \Phi_{\mathfrak{p}}(\sigma)P$. Theorem 4(a) tells us that $Q \equiv O \pmod{\bar{\mathfrak{p}}}$, so Q is in the kernel of reduction modulo $\bar{\mathfrak{p}}$. Further, the assumption that P is nontorsion and Theorem 4(b) imply that $Q \neq O$. Let \bar{v} be the absolute value associated to $\bar{\mathfrak{p}}$. Then on a minimal Weierstrass equation, we have $3\bar{v}(x(Q)) = 2\bar{v}(y(Q)) < 0$, so we can apply Theorem 2(c) to conclude that

$$\lambda_{\bar{\mathfrak{p}}}(Q) = \bar{v}(x(Q)/y(Q)) > 0.$$

Finally, we use the fact that for any extension L/K , the minimum positive value of \bar{v} restricted to L is $(\log p)/e$, where e is the ramification index of $\bar{\mathfrak{p}}$ in L . (See Remark 3.) \square

6. The ramified case

In this section we prove the basic estimate needed to handle the case of ramified extensions. We begin by recalling the proof of the key number field lemma, which is due to Amoroso and Dvornicich. We then apply this lemma to obtain an analogous result for elliptic curves. In [4], this was done by altering the given point to make sure it is nonzero modulo \mathfrak{P} . We describe an alternative approach in which we force the point to be zero modulo \mathfrak{P} . This allows us to work entirely within the formal group, where computations are much easier.

6.1. A congruence in ramified abelian extensions

The following lemma is modeled after [1, Lemma 2]. See also [2, Lemma 3.2]; [4, Lemma 3.5].

Lemma 6 (Amoroso-Dvornicich [1]). *Let K/\mathbb{Q} be a number field, let \mathfrak{p} be a degree 1 prime in K with residue characteristic p , and let L/K be an abelian extension that is ramified at \mathfrak{p} . Let \mathfrak{P} be a prime of L lying over \mathfrak{p} , and let $\mathcal{O}_{L,\mathfrak{P}}$ denote the localization of L at \mathfrak{P} . Then there exists an element $\tau \in I_{L/K}$ with $\tau \neq 1$ such that*

$$\tau(\alpha)^p \equiv \alpha^p \pmod{p\mathcal{O}_{L,\mathfrak{P}}} \quad \text{for all } \alpha \in \mathcal{O}_{L,\mathfrak{P}}.$$

(Note that the strength of this result is that the congruence is modulo p , and not merely modulo \mathfrak{P} .)

Proof. Without loss of generality, we may replace K and L by their completions $K_{\mathfrak{p}} = \mathbb{Q}_p$ and $L_{\mathfrak{P}}$, respectively. Then L is an abelian extension of \mathbb{Q}_p , so by the local Kronecker–Weber theorem, there is an integer $m \geq 1$ so that $L \subset \mathbb{Q}_p(\zeta_m)$. (Here ζ_m is a primitive m th root of unity.) We take m to be minimal, i.e., m is the conductor of L/\mathbb{Q}_p .

The extension L/\mathbb{Q}_p is ramified by assumption, which implies that $p|m$. Let τ be a generator for the cyclic group $G_{\mathbb{Q}_p(\zeta_m)/\mathbb{Q}_p(\zeta_{m/p})}$. The minimality of m implies that $L \not\subset \mathbb{Q}_p(\zeta_{m/p})$, so the restriction of τ to $G_{L/K}$ is not trivial. Further, since τ fixes ζ_m^p , we see that

$$\tau(\zeta_m) = \omega \zeta_m \quad \text{for some primitive } p\text{th root of unity } \omega.$$

Now let $\alpha \in \mathcal{O}_L \subset \mathcal{O}_{\mathbb{Q}_p(\zeta_m)} = \mathbb{Z}_p[\zeta_m]$. Thus $\alpha = f(\zeta_m)$ for some polynomial $f(X) \in \mathbb{Z}_p[X]$, and hence

$$\tau\alpha = \tau(f(\zeta_m)) = f(\tau\zeta_m) = f(\omega\zeta_m).$$

Now taking p th powers yields

$$\begin{aligned} (\tau\alpha)^p &= f(\omega\zeta_m)^p \equiv f((\omega\zeta_m)^p) \pmod{p\mathbb{Z}_p[\zeta_m]} \\ &= f(\zeta_m^p) \\ &\equiv f(\zeta_m)^p \pmod{p\mathbb{Z}_p[\zeta_m]} \\ &= \alpha^p. \end{aligned}$$

Since $L \cap p\mathbb{Z}_p[\zeta_m] = p\mathcal{O}_L$, this completes the proof. \square

6.2. Ramified points in $E(K^{\text{ab}})$

In this section we prove the key estimate needed to handle the case of ramified extensions. In particular, by applying $(\tau - 1)^2$ to a point, we can do most of our computations in the formal group, where the multiplication-by- p map is easier to describe.

Lemma 7. Let $E/K_{\mathfrak{p}}$ be an elliptic curve defined over a local field $K_{\mathfrak{p}}/\mathbb{Q}_p$ and assume that E has good reduction at \mathfrak{p} . Let $L_{\mathfrak{P}}/K_{\mathfrak{p}}$ be a finite Galois extension that is ramified at \mathfrak{p} , let $\mathcal{O}_{\mathfrak{P}}$ be the ring of integers of $L_{\mathfrak{P}}$, and let $\tau \in I_{L_{\mathfrak{P}}/K_{\mathfrak{p}}}$ be in the inertia group of $L_{\mathfrak{P}}/K_{\mathfrak{p}}$. We denote by $E_1(L_{\mathfrak{P}})$ the kernel of reduction and by \hat{E} the formal group of E , so there is a natural isomorphism $E_1(L_{\mathfrak{P}}) \cong \hat{E}(\mathfrak{P})$ [27, Chapter IV].

- (a) Let $P \in E(L_{\mathfrak{P}})$. Then $(\tau - 1)P \in E_1(L_{\mathfrak{P}})$.
 (b) Fix a minimal Weierstrass equation for E and let $z = -x/y$ be a parameter for the formal group \hat{E} . Then

$$z([p](\tau - 1)Q) \in ((\tau z(Q))^p - z(Q)^p)\mathcal{O}_{\mathfrak{P}} + p\mathcal{O}_{\mathfrak{P}}$$

$$\text{for all } Q \in E_1(L_{\mathfrak{P}}) \cong \hat{E}(\mathfrak{P}).$$

Proof. (a) This is clear, since an element of the inertia group fixes everything modulo \mathfrak{P} , so $\tau P \equiv P \pmod{\mathfrak{P}}$, and hence the difference $\tau P - P$ lies in the kernel of reduction.

(b) This is an immediate consequence of the following elementary lemma about commutative formal groups, applied with $x = \tau z(Q)$ and $y = z(Q)$. \square

Lemma 8. Let R be a ring, let $p \in \mathbb{Z}$ be a prime, and let

$$F(x, y) \in R[[x, y]]$$

be a formal group over R . Let $\iota(t) \in R[[t]]$ be the inversion series for F and let $M_p(t) \in R[[t]]$ be the multiplication-by- p series for F . Then

$$M_p(F(x, \iota(y))) \in (x^p - y^p)R[[x, y]] + pR[[x, y]].$$

Proof. There are power series $a(t), b(t) \in R[[t]]$ with $a(0) = b(0) = 0$ so that multiplication-by- p is given by power series of the form

$$M_p(t) = a(t^p) + pb(t).$$

See [27, IV.4.4]. Further, the definition of $\iota(t)$ implies that $F(x, \iota(y))$ vanishes when $y = x$, and hence that $F(x, \iota(y))$ is divisible by $x - y$, say

$$F(x, \iota(y)) = (x - y)G(x, y) \quad \text{with } G(x, y) \in R[[x, y]].$$

Then

$$M_p(F(x, \iota(y))) \in a(F(x, \iota(y))^p) + pR[[x, y]]$$

$$\in a((x - y)^p G(x, y)^p) + pR[[x, y]]$$

$$\begin{aligned} &\in (x - y)^p R[[x, y]] + pR[[x, y]] \\ &\in (x^p - y^p)R[[x, y]] + pR[[x, y]]. \quad \square \end{aligned}$$

Combining Lemmas 6 and 7 gives the following crucial local contribution to the canonical height in the ramified case. The key is the fact that the lower bound does not depend on the ramification degree of the field of definition of the point.

Proposition 9. *Let K/\mathbb{Q} be a number field, let $P \in E(K^{\text{ab}})$, and let $L = K(P)$ be the field of definition of P . Fix a degree 1 unramified prime \mathfrak{p} of K , let $p = \mathbb{N}_{K/\mathbb{Q}}\mathfrak{p}$, and suppose that \mathfrak{p} ramifies in L . Then there exists an element $\tau \in I_{L/K}$ with $\tau \neq 1$ such that the point*

$$P' = [p]((\tau - 1)^2 P)$$

satisfies

$$\hat{\lambda}_{\mathfrak{p}}(P') \geq \log p$$

for all primes \mathfrak{P} of L lying over \mathfrak{p} . (Note that if $P' = O$, then $\hat{\lambda}_{\mathfrak{p}}(P') = \infty$ by definition, so the result is vacuously true.)

Proof. The extension L/K is abelian, so Lemma 6 says that there is a nontrivial element $\tau \in I_{L/K}$ such that

$$(\tau\alpha)^p \equiv \alpha^p \pmod{p\mathcal{O}_L} \quad \text{for all } \alpha \in \mathcal{O}_L. \quad (2)$$

Let $Q = (\tau - 1)P$, so $P' = [p](\tau - 1)Q$. Lemma 7(a) says that Q is in the kernel of reduction modulo \mathfrak{P} . Fix a minimal Weierstrass equation for E at \mathfrak{p} and let $z = -x/y$. Note that $z(Q)$ is in the localization $\mathcal{O}_{L,\mathfrak{P}}$ of \mathcal{O}_L at \mathfrak{P} . Lemma 7 tells us that

$$z(P') \in ((\tau z(Q))^p - z(Q)^p)\mathcal{O}_{L,\mathfrak{P}} + p\mathcal{O}_{L,\mathfrak{P}}.$$

Applying (2) with $\alpha = z(Q)$, we find that $z(P') \in p\mathcal{O}_{L,\mathfrak{P}}$. Hence Theorem 2(c) yields

$$\hat{\lambda}_{\mathfrak{P}}(P') = v_{\mathfrak{P}}(z(P')) \geq v_{\mathfrak{P}}(p) = \log p. \quad \square$$

7. Proof of the main theorem

We are now ready to prove our main result, which we restate for the convenience of the reader.

Theorem 10. *Let K/\mathbb{Q} be a number field, let E/K be an elliptic curve, and let $\hat{h} : E(\bar{K}) \rightarrow \mathbb{R}$ be the canonical height on E . There is a constant $C(E/K) > 0$ such that every*

nontorsion point $P \in E(K^{\text{ab}})$ satisfies

$$\hat{h}(P) > C.$$

Proof. Baker [4] has proven Theorem 10 in the case that the elliptic curve has complex multiplication, so we may assume that $\text{End}(E/\bar{K}) = \mathbb{Z}$. (Baker also proves the theorem in the case that $j(E)$ is nonintegral, but our proof will cover all non-CM curves.)

We begin by fixing a rational prime p and a prime \mathfrak{p} of K lying over p with the following properties:

1. $E(K^{\text{ab}})[p] = \{O\}$.
2. $p \geq \exp([K : \mathbb{Q}](1 + c_1(E/K)))$, where $c_1(E/K)$ is the constant appearing in Theorem 2(b).
3. E has good reduction at \mathfrak{p} .
4. \mathfrak{p} is an unramified prime of degree 1.

Our assumption that $\text{End}(E/\bar{K}) = \mathbb{Z}$ and Serre's Theorem 3 tell us that (1) eliminates only finitely many primes p , and similarly (2) and (3) eliminate only finitely many primes \mathfrak{p} . Condition (4) describes a set of primes of density one, so we can find a prime satisfying (1)–(4) that depends only on the field K and the curve E/K . Note, however, that we are strongly using here the assumption that E does not have CM, since if E has CM, then (1) may be false for all primes.

Let $P \in E(K^{\text{ab}})$ be a nontorsion point and let $L = K(P)$ be its field of definition. The proof of the theorem proceeds by induction on the ramification degree of L/K at \mathfrak{p} .

We begin with the unramified case, say $\mathfrak{p}\mathcal{O}_L = \mathfrak{P}_1\mathfrak{P}_2 \cdots \mathfrak{P}_d$. The extension L/K is abelian, so the Frobenius elements associated to the \mathfrak{P}_i are all the same. Let $\sigma = (\mathfrak{P}_i, L/K) \in G_{L/K}$ be this Frobenius element, and let $\Phi_{\mathfrak{p}}(X) = X^2 - aX + p$ be the characteristic polynomial of Frobenius. Then Theorem 4(b) tells us that $\Phi_{\mathfrak{p}}(\sigma)P$ is a nontorsion point, so Corollary 5 and our assumption that \mathfrak{P}_i is unramified imply that

$$\hat{\lambda}_{\mathfrak{P}_i}(\Phi_{\mathfrak{p}}(\sigma)P) \geq \log p.$$

Note that this is true for every \mathfrak{P}_i , because every \mathfrak{P}_i has the same $\sigma \in G_{L/K}$ as its Frobenius element. Adding over the \mathfrak{P}_i , we obtain the estimate

$$\sum_{\mathfrak{P}|\mathfrak{p}} \frac{[L_{\mathfrak{P}} : K_{\mathfrak{p}}]}{[L : K]} \hat{\lambda}_{\mathfrak{P}}(\Phi_{\mathfrak{p}}(\sigma)P) \geq \log p. \quad (3)$$

Next we combine this positive lower bound with the (potentially negative) contribution from the other absolute values. We compute:

$$\begin{aligned}
 & \hat{h}(\Phi_p(\sigma)P) \\
 &= \sum_{w \in M_L} \frac{[L_w : \mathbb{Q}_w]}{[L : \mathbb{Q}]} \hat{\lambda}_w(\Phi_p(\sigma)P) \quad (\text{Theorem 2(a)}) \\
 &\geq \sum_{w \in M_L, w|v_p} \frac{[L_w : \mathbb{Q}_w]}{[L : \mathbb{Q}]} \hat{\lambda}_w(\Phi_p(\sigma)P) \\
 &\quad + \sum_{w \in M_L, w \nmid v_p} \frac{[L_w : \mathbb{Q}_w]}{[L : \mathbb{Q}]} \min\{\hat{\lambda}_w(\Phi_p(\sigma)P), 0\} \\
 &\geq \left(\sum_{w \in M_L, w|v_p} \frac{[L_w : \mathbb{Q}_w]}{[L : \mathbb{Q}]} \hat{\lambda}_w(\Phi_p(\sigma)P) \right) - c_1(E/K) \quad (\text{Theorem 2(b)}) \\
 &\geq \frac{\log p}{[K : \mathbb{Q}]} - c_1(E/K) \quad (\text{from (3)}) \\
 &\geq 1 \quad \text{from the choice of } p.
 \end{aligned}$$

In order to obtain a lower bound for $\hat{h}(P)$, we use the fact that \hat{h} is a Galois invariant [27, VIII.5.10] positive semidefinite quadratic form [27, VIII.9.3]. Hence

$$\begin{aligned}
 \hat{h}(\Phi_p(\sigma)P) &= \hat{h}(\sigma^2 P - [a]\sigma P + [p]P) \\
 &\leq 3(\hat{h}(\sigma^2 P) + \hat{h}([a]\sigma P) + \hat{h}([p]P)) \\
 &= 3(\hat{h}(P) + a^2 \hat{h}(P) + p^2 \hat{h}(P)) \\
 &\leq 3(1 + 4p + p^2) \hat{h}(P) \quad \text{since } |a| \leq 2\sqrt{p}.
 \end{aligned}$$

This gives the lower bound

$$\hat{h}(P) \geq \frac{1}{3(1 + 4p + p^2)},$$

and since p was chosen depending only on E/K , independent of the point $P \in E(K^{\text{ab}})$, this completes the proof of Theorem 10 in the case that the extension $K(P)/K$ is unramified at \mathfrak{p} .

Assume now that L/K is ramified at \mathfrak{p} and assume by induction that the theorem is proven for all points defined over abelian extensions whose \mathfrak{p} -ramification index is strictly smaller than $e_{\mathfrak{p}}(L/K)$. Let $\tau \in I_{\mathfrak{p}}(L/K)$ be the nontrivial element in the inertia group described in Proposition 9, so the point

$$P' = [p](\tau - 1)^2 P$$

satisfies $\hat{\lambda}_{\mathfrak{P}}(P') \geq \log p$ for all primes $\mathfrak{P} | \mathfrak{p}$. Summing over these primes, we find that

$$\sum_{\mathfrak{P} | \mathfrak{p}} \frac{[L_{\mathfrak{P}} : K_{\mathfrak{p}}]}{[L : K]} \hat{\lambda}_{\mathfrak{P}}(P') \geq \log p.$$

Note that this is exactly the same inequality that we obtained earlier (cf. (3)), except that it applies to the point P' , rather than to the point $\Phi_{\mathfrak{p}}(\sigma)P$. Hence as long as we know that $P' \neq O$, then the same computation as was done above for $\Phi_{\mathfrak{p}}(\sigma)P$ yields the same lower bound for P' , namely

$$\hat{h}(P') \geq 1.$$

In the other direction, we can estimate

$$\begin{aligned} \hat{h}(P') &= \hat{h}([p](\tau - 1)^2 P) = p^2 \hat{h}(\tau^2 P - [2]\tau P + P) \\ &\leq 3p^2 (\hat{h}(\tau^2 P) + 4\hat{h}(\tau P) + \hat{h}(P)) \\ &= 18p^2 \hat{h}(P). \end{aligned}$$

Hence $\hat{h}(P) \geq 1/(18p^2)$, which completes the proof of Theorem 10 provided $P' \neq O$.

Finally, suppose that

$$P' = [p](\tau - 1)^2 P = O.$$

Let m be the order of τ in $G_{L/K}$. There are polynomials $a(X), b(X) \in \mathbb{Z}[X]$ satisfying

$$a(X)(X^m - 1) + b(X)p(X - 1)^2 = mp(X - 1).$$

(The existence of such an identity follows immediately from the fact that the resultant of $X^{m-1} + \cdots + X + 1$ and $X - 1$ is m . Explicitly, one can take $a(X) = p$ and $b(X) = -\sum_{i=0}^{m-2} (m - 1 - i)X^i$.) We evaluate this identity at $X = \tau$ and apply it to the point P . We know that $\tau^m = 1$ and $[p](\tau - 1)^2 P = O$, so we find that

$$[mp](\tau - 1)P = O.$$

In particular, $(\tau - 1)P \in E(L)_{\text{tors}}$. However, we also know from Lemma 7 that $(\tau - 1)P$ is in the kernel of reduction modulo \mathfrak{P} , so it follows from general facts about formal groups [27, IV.3.2(b)] that the order of $(\tau - 1)P$ is a power of p . We now recall that p was chosen so that $E(K^{\text{ab}})[p] = \{O\}$, from which we conclude that $(\tau - 1)P = O$.

Thus P is fixed by τ , so P lies in the proper subfield L^{τ} of L . (Note that $\tau \neq 1$, so $L^{\tau} \neq L$.) Further, P is a nontorsion point by assumption. Hence by induction we conclude that $\hat{h}(P) \geq C(E/K)$, which completes the proof of the theorem. \square

8. Generalization to abelian varieties

Much of material in this paper can be generalized to the case of abelian varieties, at the usual cost of making the arguments more complicated. In this section we will sketch the changes required to prove the following generalization of Theorem 10.

Theorem 11. *Let K/\mathbb{Q} be a number field, let A/K be an abelian variety, let \mathcal{L} be a symmetric ample line bundle on A/K , and let*

$$\hat{h}_{\mathcal{L}} : A(\bar{K}) \rightarrow \mathbb{R}$$

be the canonical height on A associated to \mathcal{L} . Assume that A/K contains no abelian subvarieties having complex multiplication. Then there is a constant $c = c(A/K, \mathcal{L}) > 0$ such that for all nontorsion points $P \in A(K^{\text{ab}})$,

$$\hat{h}_{\mathcal{L}}(P) \geq c.$$

Proof (Sketch). Replacing K by a finite extension, we may assume that A splits into a product of geometrically simple abelian varieties, and then by looking at the individual factors, we may assume that A itself is geometrically simple. Next, replacing \mathcal{L} by $\mathcal{L}^{\otimes n}$, we may assume that \mathcal{L} is very ample. Fix effective divisors D_1, \dots, D_r for \mathcal{L} whose intersection consists of only the point O , and then fix canonical local heights (also known as Néron functions)

$$\hat{\lambda}_{D_i} : M_{\bar{K}} \times A(\bar{K}) \rightarrow \mathbb{R} \cup \{\infty\}.$$

See [12, Chapters 10 and 11] for standard properties of local and global height functions. In particular, standard properties of (canonical) local height functions give the analog of Theorem 2. Thus [12, Chapter 11, Theorem 1.6] says that for any nonzero point $Q \in A(\bar{K})$, we can compute $\hat{h}_{\mathcal{L}}(Q)$ as

$$\hat{h}_{\mathcal{L}}(Q) = \sum_{w \in M_{K(Q)}} \frac{[K(Q)_w : \mathbb{Q}_w]}{[K(Q) : \mathbb{Q}]} \hat{\lambda}_{D_i}(w, Q)$$

by choosing an index i with $Q \notin \text{Support}(D_i)$, which gives the analog of Theorem 2(a). Similarly, the analog of Theorem 2(b) follows fairly directly from the M_K -positivity of local height functions attached to positive divisors [12, Chapter 10, Proposition 3.1], and the analog of Theorem 2(c) follows from the fact that for places of good reduction, the canonical local height is given as an intersection index on the special fiber of the Néron model of A [12, Chapter 11, Theorem 5.1].

Next consider Theorem 3, which uses a deep result of Serre to limit the torsion in $E(K^{\text{ab}})$. For the proof of Theorem 11, we really only need to know that

$$A(K^{\text{ab}})[p] = 0 \quad \text{for all but finitely many primes } p. \quad (4)$$

(Indeed, it would suffice to know that $A(K^{\text{ab}})[p] = 0$ for infinitely many primes.) In certain cases, Serre has proven an “image of Galois” theorem for abelian varieties [24]. Fortunately, the weaker statement (4) that we require is proven in full generality by Zarhin [33] (see also [21]). All of these results [21,24,33] rely heavily on the groundbreaking methods used by Faltings in his proof of Tate’s Isogeny Conjecture [8].

Theorem 4 generalizes directly to abelian varieties. The characteristic polynomial $\Phi_p(X)$ of Frob_p acting on $T_\ell(A)$ is a monic polynomial of degree $2g$ whose roots have norm \sqrt{q} . See [18, Section 18] or [19, Section 21, Application II]. The proof of both parts of Theorem 4 is an easy consequence of these facts and the injectivity of

$$\text{End}(\tilde{A}/\mathbb{F}_{\tilde{p}}) \rightarrow \text{End}(T_\ell(\tilde{A})).$$

Further, if we write $\Phi_p(X) = \sum a_i X^{2g-i}$, then the triangle inequality yields $|a_i| \leq \binom{2g}{i} q^{i/2}$. This allows us to handle the unramified case.

The ramified case again relies on the Amoroso–Dvornicich Lemma 6, together with an abelian variety version of Lemma 7. And just as in the case of elliptic curves, Lemma 7 for abelian varieties follows from standard facts about commutative formal groups. See [9] or [10] for basic results on formal groups, and in particular a description of the multiplication-by- p map that gives a higher dimensional version of Lemma 8.

Having now assembled all of the required pieces, they fit together to give a proof of Theorem 11 using the same argument as given in the proof of Theorem 10. \square

Remark 6. Combining the methods of this paper, which essentially handles the non-CM case, with the method for CM elliptic curves described in Baker’s article [4], one may be able to prove in full generality the inequality $\hat{h}_{\mathcal{L}}(P) \geq C(A/K, \mathcal{L})$ for all nontorsion points $P \in A(K^{\text{ab}})$ on all abelian varieties A/K . (This is essentially Conjecture 1.10 in [4].) The details of this argument will be given in a subsequent publication.

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